$$m_{12} = \pi R_h^2 \cdot \pi^2/12 \quad (R_h = (^1/_2 R^{-1} + ^1/_2 r^{-1})^{-1})$$

I am grateful to G. S. Kulikov for drawing this problem to my attention.

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SIMPLE WAVES IN PRANDTL–REUSS EQUATIONS†

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The solution of the system of equations of plane simple waves in a Prandtl–Reuss isotropically work-hardening medium is reduced in general (without any assumptions on the form of the work-hardening function and the state in front of the simple wave) to the investigation of an ordinary differential equation of the first order. In the special case of linear work-hardening, and also without work-hardening, the solution of the system of equations for plane simple waves is obtained in quadratures. The problem of an oblique shock on a prestressed half-space with arbitrary uniform constant stresses is solved for a linearly work-hardening medium.

FOR THE Prandtl–Reuss equations, the corresponding system of ordinary differential equations of plane simple waves sometimes splits (because the component equations are uncoupled) and thus admits of a straightforward analysis. Plane simple waves propagating along the x^1 axis of the

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Cartesian system of coordinates x_i have been studied [1–4] for the case $v_3 = 0$, $\sigma_{13} = 0$ (v_i and σ_{ij} are the velocity and stress components). The investigation of these waves has been reduced to quadratures (an ideal elastoplastic medium with the supplementary condition $\sigma_{22} - \sigma_{33} = 0$ [1], a linearly work-hardening medium with $\sigma_{22} = \sigma_{33} = 0$ [2]) or to the analysis of an ordinary differential equation—numerical for an isotropically work-hardening medium with $\sigma_{22} - \sigma_{33} = 0$ [3] or qualitative for an ideal elastoplastic medium with $\sigma_{22} - \sigma_{33} \neq 0$ [4].

Unlike previous studies [1–4], the present investigation does not impose any restrictions on the form of the initial stressed state of the half-space. The proposed method produces a solution using only three families of plane curves. Numerical solution of the ordinary differential equations of simple waves is thus avoided.

1. REDUCTION TO AN ANALYSIS OF AN ORDINARY DIFFERENTIAL EQUATION

We consider the plane-wave motion of an elastoplastic medium in the framework of geometrically linear theory. The loading surface equation is assumed in the Mises form and the work-hardening parameter is the plastic strain work (the superscript d denotes the tensor deviator and ε_{ij}^p are the components of the plastic strain tensor)

$$^1/_2\sigma_{ij}^d\sigma_{ij}^d = f(\chi) \quad (d\chi = \sigma_{ij}d\varepsilon_{ij}^p) \tag{1.1}$$

When condition (1.1) is satisfied, we accept the associated law $d\varepsilon_{ij}^p = d\lambda\sigma_{ij}^d$, $d\lambda \geqslant 0$ and Hooke's law for elastic strains. For this medium, the system of plane-wave equations has the form (μ, K are the constant moduli of elasticity)

$$\begin{gathered}\rho_0\frac{\partial v_i}{\partial t} = \frac{\partial\sigma_{i1}}{\partial x}, \quad \frac{\partial\sigma_{kk}}{\partial t} = 3K\frac{\partial v_1}{\partial x}\\ e_{ij}^d = \frac{1}{2\mu}\frac{\partial\sigma_{ij}^d}{\partial t} + \frac{\partial\lambda}{\partial t}\sigma_{ij}^d; \quad \frac{\partial\lambda}{\partial t} \geqslant 0\\ ^1/_2\sigma_{ij}^d\sigma_{ij}^d = f(\chi), \quad \frac{\partial\chi}{\partial t} = \frac{\partial\lambda}{\partial t}2f(\chi)\end{gathered} \tag{1.2}$$

If the front of the wave $\sigma_{23} = 0$, $\sigma_{22} - \sigma_{33} = \gamma_0 = \text{const}$ (the first equality always can be achieved by rotating the system of coordinates about the x^1 axis), then from the third relationship in system (1.2) we obtain

$$\sigma_{23} = 0, \quad \sigma_{22} - \sigma_{33} = \gamma_0 e^{-2\mu\lambda} \tag{1.3}$$

In what follows, we assume that these relationships are satisfied.

Let us investigate the simple waves of the system (1.2), i.e. solutions of the form

$$\sigma_{ij}(\theta(x, t)), \quad v_i(\theta(x, t)), \quad \lambda(\theta(x, t)), \quad \chi(\theta(x, t)), \quad x \equiv x^1$$

The system of ordinary differential equations describing the simple waves has the form (prime denotes the derivatives with respect to θ)

$$\begin{gathered}v_1'(K + {}^4/_3\mu - p) = -2\mu\sigma_{11}^d c\lambda'\\ v_j'(\mu - p) = -2\mu\sigma_{1j}c\lambda' \quad (j = 2, 3)\\ \sigma_{kk}'(K + {}^4/_3\mu - p) = 6K\mu\sigma_{11}^d\lambda'\\ \sigma_{11}^{d\prime}(K + {}^4/_3\mu - p) = 2\mu(p - K)\sigma_{11}^d\lambda'\\ \sigma_{1j}'(\mu - p) = 2\mu\sigma_{1j}p\lambda' \quad (j = 2, 3)\\ {}^3/_4(\sigma_{11}^d)^2 + \sigma_{12}^2 + \sigma_{13}^2 + {}^1/_4(\sigma_{22} - \sigma_{33})^2 = f(\chi), \quad \chi' = 2f\lambda'\end{gathered} \tag{1.4}$$

Here $c = -\theta_t'/\theta_x'$ is the characteristic velocity and $p = \rho_0 c^2$ is obtained from the characteristic equation

$$(K + {}^4/_3\mu - p)(\mu - p) - \frac{2\mu^3}{(2\mu + f')f}\{(\mu - p)(\sigma_{11}{}^d)^2 +$$
$$+ (K + {}^4/_3\mu - p)(\sigma_{12}{}^2 + \sigma_{13}{}^2)\} = 0 \qquad (1.5)$$

System (1.4) has a number of obvious first integrals. Specifically, the fifth and the second equations, respectively, give $d\sigma_{13}/d\sigma_{12} = \sigma_{13}/\sigma_{12}$, $dv_3/dv_2 = \sigma_{13}/\sigma_{12}$, whence we obtain by integrating

$$\sigma_{13} = \alpha_0\sigma_{12}, \quad v_3 = \alpha_0(v_2 - v_2{}^\circ) + v_3{}^\circ, \quad \alpha_0 = \sigma_{13}{}^\circ/\sigma_{12}{}^\circ \qquad (1.6)$$

where the constants $v_2{}^\circ$, $v_3{}^\circ$, $\sigma_{12}{}^\circ$, $\sigma_{13}{}^\circ$ are the values of the corresponding quantities in front of the wave.

The last equation in (1.4) is also integrated:

$$\lambda = \frac{1}{2}\int\frac{d\chi}{f(\chi)} = \varphi(\chi) \quad (\chi = \Psi(\lambda)) \qquad (1.7)$$

Thus, for system (1.4) there are five first integrals: (1.3), the last but one relationship in (1.4), (1.6), and (1.7).

System (1.4) can be conveniently rewritten in dimensionless variables

$$\sigma_{11}{}^* = \sigma_{11}/k_0, \quad s_{11} = \sigma_{11}{}^d/k_0, \quad s_{1j} = \sigma_{1j}/k_0, \ (j = 2, 3)$$
$$\lambda^* = 2\mu\lambda, \quad \chi^* = \chi/2\mu, \quad v_i{}^* = \sqrt{\frac{\rho_0}{\mu}}v_i \quad (i = 1, 2, 3), \quad c^* = \sqrt{\frac{\rho_0}{\mu}}c \qquad (1.8)$$

where k_0 is the initial yield point. Henceforth, the asterisks are omitted.

In these variables, the loading surface equation is written, using (1.3) and (1.7), in the form

$$^3/_4 s_{11}{}^2 + s_{12}{}^2 + s_{13}{}^2 = F(\lambda; \gamma_0) \qquad (1.9)$$

The characteristic equation (1.5) takes the form

$$(l_0 + {}^4/_3 - p)(1 - p) - \frac{2\mu k_0{}^2}{(2\mu + f')f}\{(1 - p)s_{11}{}^2 +$$
$$+ (l_0 + {}^4/_3 - p)(s_{12}{}^2 + s_{13}{}^2)\} = 0, \quad l_0 = K/\mu \qquad (1.10)$$

By (1.9), the dependence of p on $s_{12}{}^2 + s_{13}{}^2$ can be eliminated and therefore the characteristic velocity c depends only on the variables $s_{11}{}^2$, λ and the initial parameter γ_0. Then the fourth equation in (1.4) splits from the system. In fact, it takes the form

$$\frac{ds_{11}{}^2}{d\lambda} = \frac{p - l_0}{l_0 + {}^4/_3 - p}s_{11}{}^2 \equiv f_1(\lambda, s_{11}{}^2; \gamma_0) \qquad (1.11)$$

and in principle it determines the dependence ($s_{11}{}^\circ$ is the value of s_{11} for $\lambda = 0$)

$$s_{11} = s_{11}(\lambda; s_{11}{}^\circ, \gamma_0)$$

Taking this dependence into account, we rewrite the remaining differential equations of the simple-wave system (1.4) in the form

$$\frac{dv_1}{d\lambda} = -\frac{k_0}{\mu}\frac{c}{l_0 + {}^4/_3 - p}s_{11} \equiv f_2(\lambda; s_{11}{}^\circ, \gamma_0)$$
$$\frac{dv_2}{d\lambda} = -\frac{k_0}{\mu}\frac{c}{1 - p}s_{12} \equiv f_3(\lambda; s_{11}{}^\circ, \gamma_0)$$
$$\frac{d\sigma_{11}}{d\lambda} = \frac{p}{l_0 + {}^4/_3 - p}s_{11} \equiv f_4(\lambda; s_{11}{}^\circ, \gamma_0)$$

From these equations we obtain v_1, v_2 and σ_{11} by quadratures; the variables s_{12}, s_{13} and v_3 are determined from the final relationships (1.6) and (1.9).

Thus, the investigation of the system of simple-wave Prandtl–Reuss equations reduces to the solution of a first-order differential equation and the evaluation of the integrals that depend on this equation. The result is true without any assumptions regarding the form of the work-hardening function $f(\chi)$ in front of the simple wave. Our proposition can be used to simplify the numerical analysis of simple waves, e.g. in the problem of the breakup of an arbitrary shock.

2. REDUCTION OF SIMPLE WAVE EQUATIONS TO QUADRATURES FOR LINEAR WORK-HARDENING

For a linearly work-hardening material, the function $f(\chi)$ in (1.1) has the form $f(\chi) = k_0^2 + \beta\cdot\chi$, where k_0 is the shear yield point and $\beta = \text{const} > 0$. Consider the simple wave propagating in this medium in a uniform stressed state with the stresses σ_{11}°, σ_{12}°, σ_{13}°, σ_{22}°, σ_{33}°. Such simple waves are used in what follows to solve the problem of an oblique shock in a half-space. We will show that the variation of all quantities in the simple wave is determined by quadratures. In particular, this assertion is true for an ideal elastoplastic medium ($\beta = 0$).

Using formulas (1.8), we introduce the dimensionless variables s_{1i}, v_i ($i = 1, 2, 3$), σ_{11}, χ, λ, and also the variables

$$r = (\sigma_{11}^d)^2/f, \quad \tau = (\sigma_{12}^2 + \sigma_{13}^2)/f \tag{2.1}$$

We will show that if r or τ is chosen as the simple-wave parameter, then the investigation of the simple wave reduces to quadratures. Indeed, from the last three relationships (1.4) it follows that the variables r and τ are related by the equation

$$\frac{dr}{d\tau} = \left(\frac{p - l_0}{l_0 + {}^4/_3 - p} - a\right)\left(\frac{p}{1 - p} - a\right)^{-1}\frac{r}{\tau} = \Phi(r, \tau), \quad a = \frac{\beta}{2\mu} \tag{2.2}$$

where p is the root of the characteristic equation

$$(l_0 + {}^4/_3 - p)(1 - p) - b(1 - p)r - b(l_0 + {}^4/_3 - p)\tau = 0, \qquad b = 2\mu/(2\mu + \beta) \tag{2.3}$$

Let $p_i = F_i(r, \tau)$ ($i = 1, 2$) be the roots of Eq. (2.3). Solving these relationships for r and τ, we make the following change of variables in Eq. (2.2):

$$\begin{aligned}(l_0 + {}^1/_3)\, br &= (l_0 + {}^4/_3 - p_1)(l_0 + {}^4/_3 - p_2)\\ (l_0 + {}^1/_3)\, b\tau &= -(1 - p_1)(1 - p_2)\end{aligned} \tag{2.4}$$

It thus takes the form of a Riccati equation

$$\begin{aligned}\{a(l_0 + {}^1/_3) + l_0\}\, dp_1/dp_2 = {}&\{(1 + a)p_1 - l_0 - a(l_0 + {}^4/_3)\}(1 - p_1)/(1 - \\ &- p_2) - \{(1 + a)p_1 - a\}(l_0 + {}^4/_3 - p_1)/(l_0 + {}^4/_3 - p_2)\end{aligned} \tag{2.5}$$

This equation has been written for fast simple waves: p in Eq. (2.2) has been replaced with the large root p_1 of Eq. (2.3); slow simple waves are investigated similarly.

In this case, a particular solution of Eq. (2.5) has the form

$${}^3/_4(l_0 + {}^4/_3 - p_1)(l_0 + {}^4/_3 - p_2) - (1 - p_1)(1 - p_2) = (l_0 + {}^1/_3)b$$

and it can be used to obtain the general solution of this equation:

$$p_1 = \left\{\int_{p_2^\circ}^{p_2} b(t)\varphi(t)\,dt + a_0\varphi(p_2^\circ)\right\}^{-1}\varphi(p_2) + \Psi_0(p_2)$$

$$b(t) = \frac{(1 + a)(l_0 + {}^1/_3)}{m(1 - t)(l_0 + {}^4/_3 - t)}, \quad \Psi_0(p_2) = 3l_0\frac{l_0 + {}^4/_3 + a_1 - p_2}{p_2 + l_0}$$

$$\varphi(t) = \{(1 - t)^{m-1}(l_0 + {}^4/_3 - t)^{m+4/3}(t + 3l_0)^{-2m}\}^{1/m}$$

$$m = a(l_0 + {}^1/_3) + l_0, \quad a_0 = (p_1^\circ - \psi_0(p_2^\circ))^{-1}, \quad a_1 = {}^4/_3(m - l_0)/(m + a/3)$$

(p_1° and p_2° are the initial values of p_1 and p_2).

This solution of Eq. (2.5) and relationships (2.4) produce the dependence

$$r = r\,(\tau;\, r_0, \tau_0) \quad (r_0 = (s_{11}^\circ)^2, \quad \tau_0 = (s_{12}^\circ)^2 + (s_{13}^\circ)^2)$$

Henceforth we will assume that this dependence is known.

We will show that the variables s_{12}, s_{13}, γ, λ, χ are expressible in terms of r and τ. Indeed, integrating (1.7) we obtain

$$\lambda = \mu\beta^{-1} \ln\,(f\,(\chi)\, k_0^{-2}) \tag{2.6}$$

Solving the last but one equation in (1.4) for $f(\chi)$ and using (1.3) and (2.6), we obtain

$$k_0^2 + \beta 2\mu\chi = k_0^2\Lambda\,(r,\tau), \quad \Lambda\,(r,\tau) = ({}^1/_2\gamma_0/k_0)^{2(1-b)}\,(1 - {}^3/_4 r - \tau)^{-(1-b)} \tag{2.7}$$

Substituting (2.7) and (1.6) into Eq. (2.1) for the variable τ, we obtain

$$s_{12}^2 = k_0^2\,(1 + \alpha_0^2)^{-1}\,\tau\Lambda\,(r,\tau)$$

From the first integrals (1.3), (2.6) and (2.7), we have

$$\gamma^2 = \gamma_0^2\,(\Lambda\,(r,\tau))^{-a}$$

It remains to express the variables σ_{11}, v_1 and v_2 in terms of r and τ. From the third, fourth, and fifth equations in (1.4) and relationship (2.7) we obtain, for instance

$$\begin{aligned} \sigma_{11} - \sigma_{11}^\circ &= ({}^1/_2\gamma_0/k_0)^{1-b}\int_{\tau_0}^{\tau} \Psi_1(r\,(t;\, r_0, \tau_0), t)\,dt \\ \Psi_1(r,\tau) &= \operatorname{sgn}\,(s_{11}^\circ)\,\frac{p\,(1-p)\,(1 - {}^3/_4 r - \tau)^{-{}^1/_2(1-b)}\sqrt{r}}{2\,(l_0 + {}^4/_3 - p)\,((1+a)\,p - a)\,\tau} \end{aligned} \tag{2.8}$$

The variables v_1 and v_2 are similarly expressed in the form of quadratures.

Thus, in a linearly work-hardening medium, the variation of all quantities in the simple wave is expressed in quadratures.

3. INTEGRAL CURVES OF SIMPLE WAVES

To solve the oblique shock problem, we need to know, for any initial state, the behaviour of the projection of the simple-wave integral curves on the plane r, τ; σ_{11}, $s = \sqrt{s_{12}^2 + s_{13}^2}$; σ_{11}, τ. The projections of the integral curves on the r, τ plane are obtained by solving Eq. (2.2), which has been reduced to quadratures in Sec. 2. Consider the projections of the integral curves of the slow simple waves (the projections of the integral curves of the fast simple waves are constructed similarly). They form a one-parameter family of curves (with the parameters r_0^1—the coordinate of the intersection of the integral curve with the r-axis).

Figure 1 plots the results of calculations for $2\mu = 1.54 \times 10^5$ mPa, $3K = 5 \times 10^5$ mPa, $\beta = 1.1 \times 10^4$

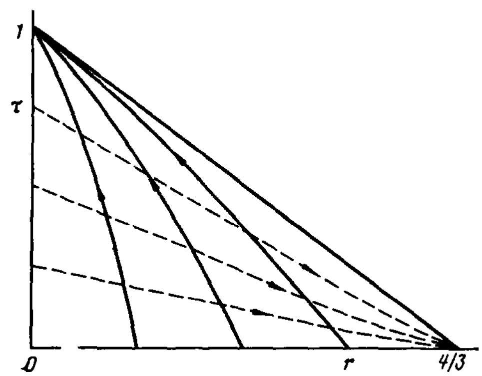

FIG. 1.

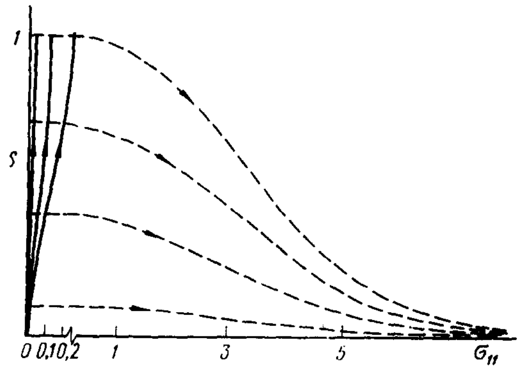

FIG. 2.

mPa, $k_0 = 450$ mPa; in all the figures, the dashed curves correspond to fast simple waves and the solid curves to slow simple waves. The direction of change of the quantities in the simple waves is shown by arrows. It is determined by the condition of activity of plastic loading $\partial\lambda/\partial t \geqslant 0$.

For a slow simple wave, the dependence of σ_{11} and s on the variable τ and the initial values σ_{11}°, r_0, τ_0 is determined, respectively, from (2.8), (2.1) and (2.7) in the form

$$\sigma_{11} - \sigma_{11}^\circ = ({}^1/_2\gamma_0(r_0, \tau_0)/k_0)^{1-b}[F_1(\tau; r_0^1(r_0, \tau_0)) - F_1(\tau_0; r_0^1(r_0, \tau_0))]$$

$$\tau \in [0, 1] \tag{3.1}$$

$$s = ({}^1/_2\gamma_0(r_0, \tau_0)/k_0)^{1-b}|1 - {}^3/_4 r(\tau; r_0^1(r_0, \tau_0), 0) - \tau|^{-\frac{1}{2}(1-b)}\tau^{\frac{1}{2}}$$

$$F_1(\tau; r_0^1(r_0, \tau_0)) = \int_0^\tau \Psi_1(r(t; r_0^1(r_0, \tau_0), 0), t)\,dt$$

In these relationships, the function $r_0^1(r_0, \tau_0)$ is determined using the family of curves shown in Fig. 1 (r_0^1 is the coordinate of the point of intersection of the curve passing through the point r_0, τ_0 with the r axis). The function $\gamma_0(r_0, \tau_0)$ is determined from the last but one relationship in (1.4).

By relationships (3.1), the projections of the simple-wave integral curves on the σ_{11}, s plane form a three-parameter family of curves with the parameters σ_{11}°, r_0, τ_0. However, the dependence on some of the parameters is quite simple. We will show that any curve from this family is in fact constructed from a one-parameter family of curves.

We will only consider the case when $\operatorname{sgn} s_{11}^\circ > 0$, because by (2.8) the projections of the simple-wave integral curves on the σ_{11}, s plane are symmetrical about the s axis for $\operatorname{sgn} s_{11}^\circ > 0$ and $\operatorname{sgn} s_{11}^\circ < 0$.

Consider the one-parameter family of curves (with the parameter r_0^1) shown by solid curves in Fig. 2 (for $\beta = 0$) and Fig. 3 (for $\alpha > 0$):

$$\sigma_{11} = F(\tau; r_0^1), \quad s = [1 - {}^3/_4 r(\tau; r_0^1, 0) - \tau]^{-\frac{1}{2}(1-b)}\tau^{\frac{1}{2}} \tag{3.2}$$

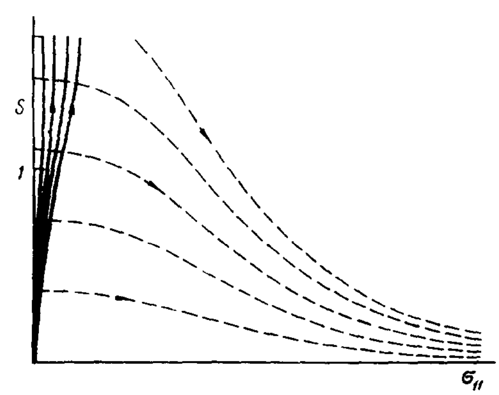

FIG. 3.

The change in s is bounded by unity for $\beta = 0$ [by the last but one relationship in (1.4)] and is unbounded for $\beta > 0$. The curves in Fig. 3 have vertical asymptotes. The right-most curves in Figs 2 and 3 correspond to $r_0^1 = 4/3$; the left-most curves (those that merge with the s axis) correspond to $r_0^1 = 0$. For $\beta = 0$ and practical values of τ_0 the range of σ_{11} is of the order of 0.25.

By virtue of relationship (3.1), the projections of the integral curves of the slow simple waves on the σ_{11}, s plane are obtained from the family of curves (3.2) by a sequence of transformations: a shift along the σ_{11} axis by a distance $-F(\tau_0; r_0^1)$, a homothety centred at zero with the coefficient $(\frac{1}{2}\gamma_0/k_0)^{1-b}$ (for $\beta = 0$, this homothety degenerates into the identity transformation), and a shift along the σ_{11} axis by a distance σ_{11}°.

Relationships (2.2), (2.8) and (3.1) lead to the following properties of the slow simple wave. The quantities σ_{11}, s vary monotonically in the slow simple wave and $\Delta\sigma_{11} > 0$ and is bounded; $\Delta s > 0$ and for $\beta = 0$ it is also bounded, while for $\beta > 0$ it increases without limit.

For the fast simple wave, we obtain from the third and fourth relations in (1.4) and equality (2.7) (it is better to use the variable r as the parameter for fast simple waves)

$$\sigma_{11} - \sigma_{11}^\circ = ({}^1/_2\gamma_0(r_0, \tau_0)/k_0)^{1-b}\,[F_2(r; \tau_0^1(r_0, \tau_0)) - F_2(r_0; \tau_0^1(r_0, \tau_0))], \quad r \in [0, {}^4/_3]$$

$$F_2(r, \tau_0^1(r_0, \tau_0)) = \int_0^r \Psi_2(t, \tau(t; \tau_0^1(r_0, \tau_0), 0))\,dt$$

$$\Psi_2(r, \tau) = {}^1/_2 \frac{p(r, \tau)}{(1+a)p - l_0(1+a) - {}^4/_3 a} (1 - {}^3/_4 r - \tau)^{-{}^1/_2(1-b)}\, r^{1/2}$$

$\tau_0^1(r_0, \tau_0)$ is the coordinate of the point of intersection of the integral curve of Eq. (2.2) through the point r_0, τ_0 with the τ axis.

The one-parameter family of curves (with the parameter τ_0^1) necessary to obtain the projections of the integral curves of the fast simple waves on the σ_{11}, s plane is given by

$$\sigma_{11} = F_2(r; \tau_0^1), \quad s = [1 - {}^3/_4 r - \tau(r; \tau_0^1, 0)]^{-1/2(1-b)} \tau^{1/2}(r; \tau_0^1, 0) \tag{3.3}$$

This family is shown by the dashed curves in Figs 2 and 3.

In the fast simple wave (like the slow wave), σ_{11} and s vary monotonically, $\Delta\sigma_{11} > 0$ and increases without limit, while $\Delta s < 0$ and is bounded ($s \to 0$ as $r \to 4/3$).

The projections of the integral curves of the slow simple waves on the σ_{11}, τ plane are obtained similarly, because in the σ_{11}, s plane the one-parameter family (3.2) gives the three-parameter family (3.1). These projections are obtained from the one-parameter family of curves (with the parameter r_0^1) defined by the first relationship in (3.2) for a composition of two parallel translations and stretching. The projections of the integral curves of the fast simple waves are similarly obtained from the one-parameter family of curves (with the parameter τ_0^1) defined by the first relationship in (3.3) and the known relation $r = r(\tau; \tau_0^1, 0)$.

These two one-parameter families of curves are shown in Fig. 4 as calculated from formulas (3.2)

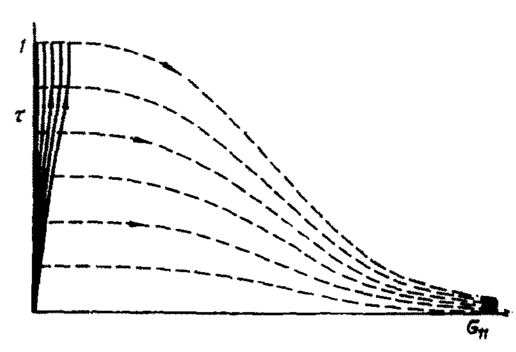

FIG. 4.

and (3.3) for $\beta>0$. For $\beta=0$, we have $s=\tau^{1/2}$, and it is therefore sufficient to construct the projections of the integral curves on the σ_{11}, s plane.

Thus, the projections of the simple-wave integral curves on the σ_{11}, s and σ_{11}, τ planes are obtained from the one-parameter families of curves shown in Figs 2–4.

4. THE PROBLEM OF OBLIQUE SHOCK ON A PRESTRESSED ELASTOPLASTIC HALF-SPACE

Let us investigate the motion of a linearly work-hardening medium (ideally plastic for $\beta=0$) with constant homogeneous initial stresses $\sigma_{ij}{}^{\circ}$ that fills the half-space $x>0$. Normal and tangential stresses $\sigma_{11}{}^{f}$, $\sigma_{12}{}^{f}$, $\sigma_{13}{}^{f}$ are applied to the surface of the half-space. These stresses arise at the instant $t=0$ and thereafter remain constant.

All the constants in Eqs (1.2) and the initial conditions of the problem have the dimensions of velocity, density, or stress. Therefore only one dimensionless combination can be formed from x and t [for instance, $xt^{-1}(\mu/\rho_0)^{-1/2}$]. The problem is self-similar.

In the half-space $x>0$, the self-similar solution consists of elastic shocks (J_1 is the longitudinal wave and J_2 is the transverse wave) and simple plastic waves (S_1 is the fast simple wave and S_2 is the slow simple wave) that propagate from left to right; these waves are separated by regions in which all the parameters are constant. The propagation sequence $J_1S_1J_2S_2$ is established by Mandel's theorem [5].

We will show how to solve the oblique shock problem using the portraits of the simple-wave integral curves from Sec. 3. We use the dimensionless variables introduced in Sec. 2. For the elastic waves J_1 and J_2, the relevant quantities, as we know, change as follows:

for the longitudinal wave

$$\Delta\sigma_{11}=-\mu(k_0)^{-1}c\Delta v_1,\quad \Delta\sigma_{22}=\Delta\sigma_{33}=\mu(k_0)^{-1}\frac{-3K+2\mu}{3K+4\mu}c\Delta v_1$$
$$\Delta v_2=\Delta v_3=\Delta s_{12}=\Delta s_{13}=0,\quad c^2=l_0+{}^4/_3 \tag{4.1}$$

for the transverse wave

$$\Delta s_{12}=-\mu(k_0)^{-1}\Delta v_2,\quad \Delta s_{13}=-\mu(k_0)^{-1}\Delta v_3 \tag{4.2}$$
$$\Delta v_1=\Delta\sigma_{11}=\Delta\sigma_{22}=\Delta\sigma_{33}=0$$

The solution of the oblique shock problem is a curve Γ in the space σ_{ij} that joins the initial point $\sigma_{ij}{}^{\circ}$ with the point whose three coordinates $\sigma_{1i}{}^{f}$ ($i=1, 2, 3$) are equal. The curve Γ consists of sections produced by the shocks J_1 and J_2, which are defined by (4.1) and (4.2), and sections described by the plastic simple waves, i.e. integral curves of the system of ordinary differential equations (1.4). The order of the curves forming Γ is determined by the propagation sequences $J_1S_1J_2S_2$.

We have shown in Sec. 2 that the variation of all quantities in the plastic simple wave is known if we know the trajectory of variation of r, τ, i.e. the integral curve of Eq. (2.2). By formulas (4.1) and (4.2), the variation of all quantities in the elastic shock waves J_1 and J_2 is expressible in terms of the increments of the variables r and τ, respectively, (the increment of s_{12} in the wave J_2 must be specified separately). Therefore, the required curve Γ is determined by its projection on the r, τ plane. The solution is thus constructed in the r, τ plane.

In order to satisfy the boundary conditions of the problem, it is helpful also to use the σ_{11}, s plane. Indeed, if the projection on the σ_{11}, s plane of some curve Γ of the form shown above joins the initial point $\sigma_{11}{}^{\circ}$, $s^{\circ}=((s_{12}{}^{\circ})^2+(s_{13}{}^{\circ})^2)^{1/2}$ with the final point $\sigma_{11}{}^{f}$, $s^{f}=((s_{12}{}^{f})^2+(s_{13}{}^{f})^2)^{1/2}$, then this curve Γ is a solution.

We will first show that if at the final point of the solution $s=s^{f}$, then a correct choice of the variation of s_{12} in the shock J_2 ensures $s_{12}=s_{12}{}^{f}$, $s_{13}=s_{13}{}^{f}$ at the final point. Indeed, if $s=s_1$ behind

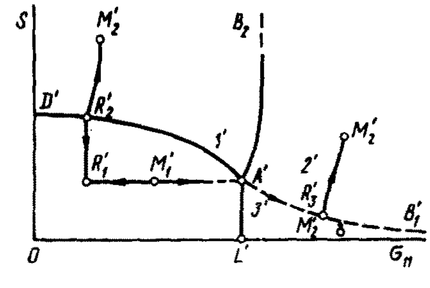

FIG. 5.

the wave S_1, then any point of the circle $s_{12}^2+s_{13}^2=s_1^2$ is reachable in the wave J_2 in the s_{12}, s_{13} plane, including the pair of points $N^i=(s_{12}^i, s_{13}^i)$ $(i=1, 2)$ defined by the condition $s_{12}^i/s_{13}^i = s_{12}^f/s_{13}^f$. Uniqueness of the shock J_2 is ensured by the condition $\operatorname{sgn} s_{12}^i = \operatorname{sgn} s_{12}^f$. In this case, by (1.6), we have at the final point of the solution $s_{12}=s_{12}^f$, $s_{13}=s_{13}^f$.

By the last but one relation in (1.4) and formulas (2.1), (4.1) and (4.2), the projection of the loading surface on the σ_{11}, s plane has the form

$$\frac{n_1^2}{12}\left(\sigma_{11}-\frac{n_2}{n_1}\right)^2+s^2=1-{}^1/_4(\sigma_{22}^\circ-\sigma_{33}^\circ)^2, \quad s\geqslant 0$$
$$n_1=\frac{4}{l_0+{}^4/_3}, \quad n_2=2\frac{{}^2/_3-l_0}{l_0+{}^4/_3}\sigma_{11}^\circ+\sigma_{22}^\circ+\sigma_{33}^\circ \qquad (4.3)$$

Note that without loss of generality we may take $n_2=0$.

Indeed, from relationships (1.4) and (4.1)–(4.3) it follows that the variation of σ_{11}^f, σ_{11}°, σ_{22}°, σ_{33}° for $\sigma_{11}^f-\sigma_{11}^\circ=\text{const}$, $r_0=\text{const}$, $\gamma_0=\text{const}$ does not affect the solution. Therefore, if we leave $\sigma_{11}^f=\sigma_{11}^\circ$, r_0, γ_0 constant and change σ_{11}^f, σ_{11}°, σ_{22}°, σ_{33}° so that $n_2=0$, the solution remains unchanged. Also note that by symmetry [which follows from relationships (2.8) and (4.1)–(4.3)], we need only consider the case $\sigma_{11}^\circ>0$, $\sigma_{11}^f>0$.

Thus, by virtue of the above remarks, the projection of the solution on the σ_{11}, s plane lies in the first quadrant, and a part of the initial loading surface is projected into the arc $D'A'$ of the ellipse (4.3) (Fig. 5) (the point A' is where the shock J_1 reaches the loading surface).

Assume that the initial state is represented in the r, τ and σ_{11}, s planes by the point M_1, M_1', respectively, which lie inside or on the loading surfaces (Figs 5 and 6). If the point M_2' corresponding to the final state in the σ_{11}, s plane lies inside or on the boundary of the figure $D'A'L'0$, then the solution is trivial: it consists of two elastic shocks J_1 and J_2. If the point M_2' lies outside this figure, then the form of the solution is determined by the region that contains this point. Let us demonstrate this assertion.

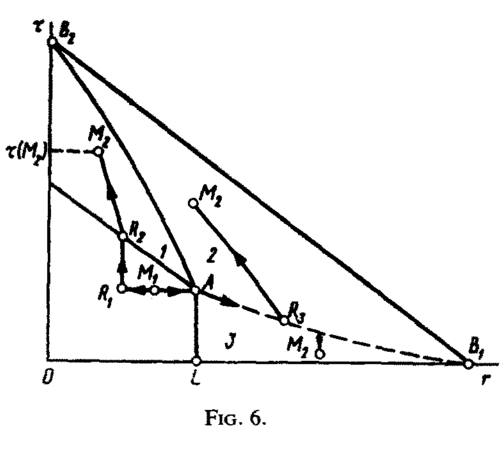

FIG. 6.

Using the family of curves shown in Figs 2 and 4, we can generate a simple wave from any point on the initial loading surface DA, as shown in Sec. 3, and construct its projections on the σ_{11}, s and σ_{11}, τ planes. From the point A, where the shock J_1 reaches the loading surface, we generate a slow and a fast simple wave, AB_1 and AB_2, respectively. The projections of these waves on the σ_{11}, s plane are $A'B_1'$ and $A'B_2'$.

The figure $DALB_1B_2$ is partitioned by the waves AB_1 and AB_2 into the regions 1–3 shown in Fig. 6. The corresponding part of the first quadrant in the σ_{11}, s plane is also partitioned by the waves $A'B_1'$ and $A'B_2'$ into the regions 1′–3′, shown in Fig. 5.

Note that if the point M_2' lies in the region 1′, then the final point of the solution M_2 in the r, τ plane lies in the region 1 and the solution has the form $J_1J_2S_2$. Similarly, if the point M_2' lies in one of the regions 2′, 3′, then the solution, correspondingly, has the form $J_1S_1J_2S_2$, $J_1S_1J_2$.

We will show how to construct the solution for the case when the point M_2' is in the region 1′. Any point R_2 on the segment AD can be reached from M_1 by elastic shocks J_1–M_1R_1 and J_1–R_1R_2. Displace the point R_2 along AD in the direction from A to D until the projection on the σ_{11}, s plane of the slow wave originating from the point R_2 reaches the point M_2'. Using the projection of this simple wave on the σ_{11}, τ plane, we obtain from the value of $\sigma_{11}{}^f$ the final value of the variable τ, which is $\tau(M_2)$. We have thus obtained the solution $M_1R_1R_2M_2$.

If the point M_2' is in the region 2′, then displace the point R_3 along the fast simple wave AB_1 in the direction from A to B_1 until the projection on the σ_{11}, s plane of the slow simple wave originating from the point R_3 reaches the point M_2'. In this case, the solution is $M_1AR_3M_2$. For the case when the point M_2' is in the region 3′, the solution is constructed similarly.

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